

# Topology of the Gauge Group in Noncommutative Gauge Theory

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## Abstract

I argue that the gauge group of noncommutative gauge theory consists of maps into unitary operators on Hilbert space of the form  $u = 1 + K$  with  $K$  compact. Some implications of this proposal are outlined.

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# 1 Introduction and Apology

My talk at the Strings 2001 meeting summarized work done over the last year on the construction of D-branes as solitons in noncommutative gauge theory. This identification initially arose in a limit of large  $B$  field [1],[2],[3],[4], and was later extended to all values of  $B$  through incorporation of the noncommutative gauge field [5]. This construction sheds new light on the properties of D-branes. For example, the  $U(n)$  gauge symmetry on  $n$  coincident branes arises as a subgroup of unitary transformations on Hilbert space. In addition, the classification of D-brane charge by K-theory [6],[7] becomes evident in this description [9],[10],[8].

Since I have reviewed this material elsewhere [19], it seemed pointless to reproduce a subset of this material for the proceedings of this conference. With apologies to the organizers, I would instead like to offer some minor comments on the structure of the gauge group in noncommutative gauge theory. This material may be known by experts, but I have not seen it discussed explicitly in the literature, and it seems to clarify some otherwise confusing aspects of noncommutative gauge theory.

## 2 Topology of the Gauge Group

This note is concerned with the topology of the gauge group of noncommutative gauge field theory defined on  $\mathbf{R}^{1,p} \times \mathbf{R}^{2d}$ . The first factor refers to  $p+1$  commuting coordinates, including time. In string theory it might represent the commuting world-volume of a Dp-brane. I will work in Euclidean space but retain the notation  $\mathbf{R}^{1,p}$ . In the second factor the Weyl-Groenewold-Moyal star product is used to define a noncommutative product of functions on  $\mathbf{R}^{2d}$  in terms of a non-degenerate symplectic form  $\theta^{ij}$ ,

$$f * g(x) = e^{\frac{i}{2}\theta^{ij}\partial_i\partial'_j} f(x)g(x')|_{x^i=x'^i} \quad (1)$$

The coordinates on  $\mathbf{R}^{1,p} \times \mathbf{R}^{2d}$  are denoted by  $(y, x)$ .

The fields in such a noncommutative gauge field theory can be viewed as functions  $f(y, x)$  which are multiplied using the star product. Equivalently, they can be mapped to operators on an infinite-dimensional, separable Hilbert space  $\mathcal{H}$ :

$$f(y, x) \rightarrow \hat{O}_f(y) \quad (2)$$

using the Weyl transform. For reviews see [11],[12],[19]. The noncommutative gauge symmetry then acts as unitary transformations on  $\mathcal{H}$

$$\hat{O}_f \rightarrow U \hat{O}_f \overline{U}, \quad (3)$$

with  $U$  unitary and  $\overline{U}$  the adjoint of  $U$ . This gauge symmetry is usually referred to in the physics literature as either  $U(\infty)$  or  $U(\mathcal{H})$ . As discussed below, these two groups have well defined mathematical meanings and are definitely quite different. For example,  $U(\mathcal{H})$  is contractible by a theorem of Kuiper [14] and so has trivial topology while  $U(\infty)$  has non-trivial homotopy groups  $\pi_n$  for all positive odd integer  $n$ .

In this note I will propose a more precise definition of the gauge group and sketch a few implications and applications of this proposal. The main observation is completely elementary, but nonetheless has a number of interesting implications.

If  $\mathcal{H}$  is an infinite-dimensional, separable, complex Hilbert space then the group of all unitary operators on  $\mathcal{H}$ ,  $U(\mathcal{H})$ , has trivial topology as noted above. There are however subgroups of operators with non-trivial topology. As summarized in [20], these may be characterized as follows. Unitary operators  $u \in U(\mathcal{H})$  of the form  $u = 1 + O$  with  $O$  finite rank define a subgroup  $U(\infty)$  of  $U(\mathcal{H})$ . Clearly  $U(\infty)$  contains  $U(N)$  for all finite  $N$  and has homotopy groups determined by Bott periodicity. Other groups are defined by taking the completion of finite rank operators with respect to the  $L^p$  norm  $\|A\|_p = (Tr|A|^p)^{1/p}$ . For  $p = \infty$  we take this to be the usual operator norm  $\|A\|_\infty = \sup\{\|Ax\| \mid \|x\| = 1\}$ . This defines a sequence of groups

$$U(\infty) \subset U_1(\mathcal{H}) \subset U_2(\mathcal{H}) \subset \cdots \subset U_{\text{cpt}}(\mathcal{H}) \quad (4)$$

with elements of the form  $u = 1 + O$  with  $O$  finite rank, trace class, Hilbert-Schmidt, on up to  $O$  compact. A theorem of Palais [17] asserts that these groups all have the same homotopy type as  $U(\infty)$ .

I will define the gauge group by analogy to the standard treatment of “commutative” gauge theory, to which the noncommutative theory should reduce in the limit of vanishing non-commutativity. This definition is also supported by the form of the Seiberg-Witten map [21] and presumably could be derived from first principles by a more careful study of noncommutative gauge theory.

Let us first recall the treatment of “commutative” gauge theories in the Euclidean path integral formalism [15]. Let  $\mathcal{A}$  be the space of gauge field configurations on  $\mathbf{R}^n$ ,  $\mathcal{G}_0$  the set of

gauge transformations (maps from  $\mathbf{R}^n$  to the gauge group  $G$ ) which approach the identity at infinity and  $\mathcal{G}'$  be the set of gauge transformations which have a limit at infinity, not necessarily equal to the identity. Then the gauge orbit space which one integrates over is

$$\mathcal{C} = \mathcal{A}/\mathcal{G}_0 \quad (5)$$

and the quotient  $\mathcal{G}_\infty = \mathcal{G}'/\mathcal{G}_0$  acts on  $\mathcal{C}$  as a global symmetry group.

This structure has an obvious analog in noncommutative field theory. Let  $\hat{\mathcal{A}}$  be the space of noncommutative gauge field configurations on  $\mathbf{R}^{1,p} \times \mathbf{R}^{1+2d}$ .<sup>1</sup> The analog of the gauge group  $\mathcal{G}_0$  should consist of unitary operators  $U(y)$  on  $\mathcal{H}$  which “approach the identity at infinity”. On the noncommutative  $\mathbf{R}^{2d}$  this means we should consider unitary operators of the form  $U = 1 + K$  with  $K$  a compact operator, i.e.  $U_{\text{cpt}}(\mathcal{H})$  (recall that compact operators map under the Weyl transform to functions on  $\mathbf{R}^{2d}$  which vanish at infinity). On  $\mathbf{R}^{1,p}$  this means we take maps from  $\mathbf{R}^{1,p}$  into  $U_{\text{cpt}}(\mathcal{H})$  which approach the identity at infinity in  $\mathbf{R}^{1,p}$ , or equivalently maps from the sphere  $S^{1+p}$  into  $U_{\text{cpt}}(\mathcal{H})$ . I will denote this gauge group by  $\hat{\mathcal{G}}_0$ .

The most natural candidate for an analog of  $\mathcal{G}_\infty$  is the quotient  $\hat{\mathcal{G}}_\infty = \hat{\mathcal{G}}'/\hat{\mathcal{G}}_0$  where  $\hat{\mathcal{G}}'$  consists of maps from  $\mathbf{R}^{1+p}$  into  $U(\mathcal{H})$  which have a limit at infinity. Note that since the compact operators form a two-sided ideal in  $B(\mathcal{H})$ ,  $U_{\text{cpt}}(\mathcal{H})$  is a closed normal subgroup of  $U(\mathcal{H})$  and so  $\hat{\mathcal{G}}_\infty$  is a well defined topological group.

I thus propose that the gauge orbit space of noncommutative gauge theory should be taken to be

$$\hat{\mathcal{C}} = \hat{\mathcal{A}}/\hat{\mathcal{G}}_0 \quad (6)$$

and that the group  $\hat{\mathcal{G}}_\infty$  acts as a global symmetry group of  $\hat{\mathcal{C}}$ . The following section sketches a few implications and applications of this proposal.

### 3 Applications

The topology of the space  $\mathcal{C}$  plays an important role in many aspects of gauge theory. Below I sketch a few applications of the above proposal for  $\hat{\mathcal{C}}$  to noncommutative gauge theory, some with direct analogs in commutative gauge theory. Note that for most of these applications

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<sup>1</sup>I will not try to give a precise definition of  $\hat{\mathcal{A}}$ , the only fact that will really be needed in what follows is that  $\hat{\mathcal{A}}$  is contractible. In the Hamiltonian framework one would define the classical configuration space of finite energy gauge fields by restricting the gauge fields  $A$  to the subset of bounded operators on  $\mathcal{H}$  such that  $\int dy \text{Tr} F^2 < \infty$

$U_{\text{cpt}}(\mathcal{H})$  could be replaced with any of the groups arising through completions of finite rank operators.

### 3.1 Noncommutative Chern-Simons Theory

One can define a noncommutative generalization of Chern-Simons theory on  $\mathbf{R} \times \mathbf{R}^{2d}$  [22],[23],[24],[25]. In the path-integral formalism we identify field configurations under gauge transformations which vanish at infinity in the space-time directions. We can therefore consider the theory on  $S^1 \times \mathbf{R}^{2d}$  and demand invariance under gauge transformations which are maps from  $S^1$  to  $U_{\text{cpt}}(\mathcal{H})$ . Since these are labelled by  $\pi_1(U_{\text{cpt}}) = \mathbf{Z}$ , one might expect to derive a quantization condition on the level of the Chern-Simons theory as in the treatment of conventional Chern-Simons theory on  $\mathbf{R}^3$  with  $\pi_3(G) = \mathbf{Z}$ . Indeed, it was found in [16],[49] that there are noncommutative gauge transformations, vanishing at infinity, which change the action unless the level is quantized <sup>2</sup>. The identification of the gauge group with  $U_{\text{cpt}}(\mathcal{H})$  gives a topological explanation of the computation of [16],[49].

### 3.2 Anomalies in Noncommutative Gauge Theory

Anomalies in non-Abelian gauge theories can be given a topological interpretation [27],[31],[26],[30]. The chiral fermion determinant defines a line bundle over  $\mathcal{C}$ . In  $2n$  spacetime dimensions the obstruction to trivializing this bundle is measured by the non-torsion part of  $\pi_2(\mathcal{C}) = \pi_1(\mathcal{G}_0) = \pi_{2n+1}(G)$ . The vanishing of this obstruction is necessary for vanishing of the anomaly but not sufficient. For example,  $U(1)$  gauge theory in four dimensions with a chiral fermion content is anomalous even though  $\pi_5(U(1)) = 0$ .

These arguments should extend to noncommutative gauge theory, see for example [50] for a general discussion of anomalies in noncommutative theories. In the context described above the obstruction to defining the determinant line bundle would be measured by  $\pi_2(\hat{\mathcal{C}}) = \pi_1(\hat{\mathcal{G}}_0) = \pi_{p+2}(U_{\text{cpt}}(\mathcal{H}))$ . The latter group is isomorphic to  $\mathbf{Z}$  for  $p$  an odd integer. Note that in this case there is a topological obstruction even for noncommutative  $U(1)$  (or  $U(2)$ ) gauge theory. This result is in agreement with recent direct computations of the anomaly in noncommutative gauge theory [29],[28].

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<sup>2</sup>In [23] a quantization condition was derived for a general class of noncommutative Chern-Simons theories based on unital  $C^*$  algebras. These correspond to compact noncommutative spaces and so involve somewhat different issues.

### 3.3 Seiberg-Witten Map

The Seiberg-Witten map [13] is a map between commutative gauge fields and gauge parameters  $(A, \lambda)$  and noncommutative gauge fields and gauge parameters  $(\hat{A}, \hat{\lambda})$  which preserves gauge equivalence. It thus defines a map from  $\mathcal{C}$  to  $\hat{\mathcal{C}}$ . Following earlier work [34],[35],[39],[36], [42], this map has now been determined to all orders in the noncommutative parameter  $\theta^{ij}$  [40],[38],[43],[41].

The above proposal for the gauge group implies that the SW map is not globally well defined since  $\mathcal{C}$  and  $\hat{\mathcal{C}}$  have different topology. For example, if we compare the gauge orbit space for noncommutative and commutative  $U(1)$  gauge theory on  $\mathbf{R}^{1,1} \times \mathbf{R}^2$  we have  $\pi_2(\hat{\mathcal{G}}) = \mathbf{Z}$  while  $\pi_2(\mathcal{G})$  is trivial. Presumably this is reflected in a non-perturbative breakdown of the SW map, a possibility that was anticipated in [13].

### 3.4 D-branes and NS fivebranes

It has been proposed that D-brane charge in the presence of a non-zero  $H$  field is described by a twisted version of K-theory [6], [48],[46],[8]. For a brief introduction to some of the relevant mathematics see [47]. In [10] it was proposed that a similar framework could be used to describe D-branes as noncommutative solitons in the presence of Neveu-Schwarz Fivebranes. In particular, it was proposed that the gauge group of noncommutative gauge theory is  $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$  and that D-branes in the presence of a NS fivebrane can be constructed utilizing  $PU(\mathcal{H})$  bundles which are twisted over the  $S^3$  used to define the fivebrane,  $\int_{S^3} H = Q_5$ . This proposal can be rephrased in light of the present proposal that the noncommutative gauge group is defined in terms of  $U_{\text{cpt}}(\mathcal{H})$ .

Elements  $u \in U(H)$  act on the algebra  $\mathcal{K}$  of compact operators as automorphisms via  $K \rightarrow uK\bar{u}$  with  $K \in \mathcal{K}$ . The kernel of this map is the  $U(1) \in U(\mathcal{H})$  generated by the identity operator. This allows one to identify  $PU(\mathcal{H})$  with the group of automorphisms of  $\mathcal{K}$ . Clearly  $PU(\mathcal{H})$  also acts as automorphisms of  $U_{\text{cpt}}(\mathcal{H})$ . Thus the proposal of [10] can be rephrased as saying that in the presence of NS fivebranes one should twist the local gauge group  $U_{\text{cpt}}(\mathcal{H})$  by an element of  $\text{Aut}(\mathcal{K}) = PU(\mathcal{H})$ .

## 4 Outlook

In commutative gauge theory the distinction between  $\mathcal{G}_0$  and  $\mathcal{G}_\infty$  plays an important role in identifying the space of collective coordinates of various solitons and instantons. For example, the dyon collective coordinate of magnetic monopoles arises from the action of  $\mathcal{G}_\infty$  as does the  $SU(2)$  orientation of instantons in  $SU(2)$  gauge theory on  $\mathbf{R}^4$ . Similar considerations arise in identifying the collective coordinates of solitons and instantons in noncommutative gauge theory as has also been pointed out in sec 3.4. of [44]. Noncommutative solitons constructed in terms of projection operators have well localized Higgs and gauge fields and so one does not expect to find collective coordinates other than the translation modes. The above considerations should however play a role in the proper treatment of collective coordinates for noncommutative monopoles and instantons.

String field theory is another area where similar issues arise. Recent work [45],[33],[32],[37] has brought out a close analogy between the construction of D-branes as noncommutative solitons [1],[2],[3],[4], [5] and the construction of D-branes as solutions of open string field theory. Both of these involve the construction of projection operators in Hilbert space. As discussed in sec 3.2 of [32], in string field theory not all unitary transformations on Hilbert space act as gauge symmetries. Although the arguments in [32] are somewhat different than those given in the previous section, it seems likely that considerations similar to those used here will be useful in giving a more concrete description of the analogs of  $\mathcal{G}_0$  and  $\mathcal{G}_\infty$  in string field theory.

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